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EVERY COUNTABLE POSET IS EMBEDDABLE IN THE POSET OF UNSOLVABLE TERMS

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Abstract. In this paper we shall show that the poset of unsolvable terms is universal.

Introduction

One defect of the current model theory of the λ-calculus is that it sheds little light on the computational behaviour of terms with trivial Böhm trees, i.e., terms with Böhm tree $\perp$. This is not an inherent defect but is rather the state of the art.

In this contribution we shall pay special attention to combinators with Böhm tree $\perp$. We want to measure how close such a combinator comes to having a nontrivial Böhm tree, i.e., a head normal form. Our approach is syntactic. This approach is not philosophically motivated; it is motivated by the taste of the author. We would be pleased to see this work superceded by further refinements in the state of the model-theoretic art.

For what follows we shall assume that the reader is familiar with [1]. Since we are interested in closed terms (combinators), all terms mentioned below should be assumed to be closed unless it is otherwise obvious.

A combinator M is solvable if for any combinator N the equation

$$MN \underset{\beta}{=} N$$

is solvable by some sequence of combinators N (possibly empty). Equivalently, M is solvable if

$$MN \underset{\beta}{=} I$$

is solvable. M_1 is more solvable than M_2 if whenever $M_2N =_\beta N$ is solvable, then so also is $M_1N =_\beta N$ (but not necessarily for the same N). Equivalently, M_1 is more solvable than M_2 if $M_1N =_\beta M_2$ is solvable.

Solvability was first studied by Barendregt. This work culminated in the Barendregt–Wadsworth theorem, viz, M is solvable if and only if M has a head normal form. In this paper we shall study the more general relation M_1 is more solvable than M_2.

We shall study this relation by considering the structure of the quotient poset. We believe that the structure of this poset sheds some light on the computational behaviour of combinators without head normal forms.

The poset of unsolvables

For combinators M and N, we say M is more solvable than N (in symbols, $M \leqslant N$) if there exist combinators $N_1 \ldots N_n$ such that $MN_1 \ldots N_n =_\beta N$. $\leqslant$ is a quasi-ordering of combinators and yields a quotient poset $\mathcal{U}$. $\mathcal{U}$ has a bottom element consisting of all solvable terms and some maximal elements (K^∞). However, the structure of $\mathcal{U}$ is quite complicated as we shall show. Our main result is that every countable poset can be isomorphically embedded in $\mathcal{U}$.

We say M has n λ's (for $n = 0, 1, \ldots \infty$) if there exist $x_1, \ldots, x_n, N$ such that $M =_\beta \lambda x_1 \ldots x_n N$. Let Λ_n be the set of combinators with n λ's but not $n+1$ λ's. It is easy to see that if $M \in \Lambda_n$ is unsolvable and N is any combinator, then $MN \in \Lambda_{n \dot{-} 1}$. Thus we can draw the picture in Fig. 1 of $\mathcal{U}$.

We now turn to the proof of our main result.

Mostowski [2] exhibited a recursive universal poset, so it suffices to embed recursive posets in $\mathcal{U}$. For each n, let $\underline{n}$ be the Church numeral for n. Suppose $(\mathbb{N}, \sqsubseteq)$ is a recursive poset. Then there exists a combinator P satisfying

$$\text{(1)} \qquad P\underline{n}\underline{m} \underset{\beta}{=} \begin{cases} \underline{m} & \text{if } n \sqsubseteq m, \\ \underline{n} & \text{else;} \end{cases}$$

$$\text{(2)} \qquad PI\underline{n} \underset{\beta}{=} P\underline{1}\underline{n} \quad \text{and} \quad P\underline{n}I \underset{\beta}{=} P\underline{n}\underline{1}$$

(this last condition is needed since we work without η). Note that $P(P\underline{n}\underline{m})\underline{m} =_\beta P\underline{n}\underline{m}$.

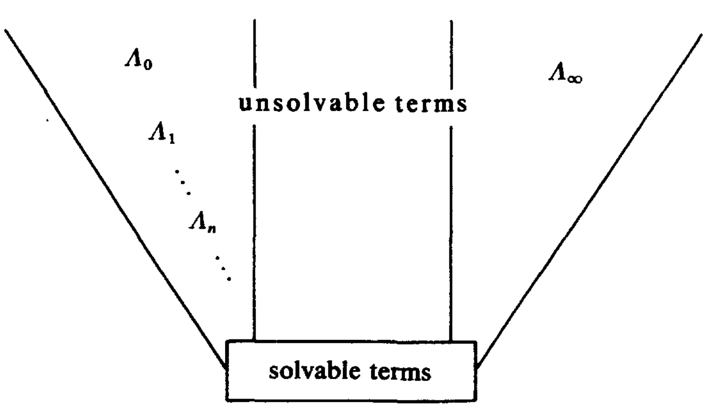

Fig. 1. The quotient poset $\mathcal{U}$.

Define terms Δ, Γ, Φ, Ψ, Ξ, E, and E_n as follows:

$$\Delta \equiv \theta\Omega, \qquad \Gamma \equiv \lambda x[x\Omega\Delta, xI], \qquad \Phi \equiv \lambda u\, u[\Delta, I],$$

$$\Psi \equiv \lambda u\, \lambda xyz\, u(Pxz)(y(\Gamma z)),$$

$$\Xi \equiv \theta\Phi, \qquad E \equiv \theta\Psi, \qquad E_n \equiv E\underline{n}\Xi,$$

where θ is defined to be Turing's fixed-point combinator and $\Omega \equiv \omega\omega \equiv (\lambda xxx)(\lambda xxx)$ as usual. We shall prove $E_n \leqslant E_m \Leftrightarrow n \sqsubseteq m$.

Observe that, for each m and k, $E_n \twoheadrightarrow$

$$\lambda z_1 \ldots z_m(E(P(\ldots(P\underline{n}z_1)\ldots)z_m)(\Xi\underbrace{[\Delta, I]\ldots[\Delta, I]}_{k}(\Gamma z_1)\ldots(\Gamma z_m))).$$

The last component of the matrix, headed by Ξ, ensures that if $E_nM_1 \ldots M_m =_\beta E_p$, then, for $1 \leqslant i \leqslant m$, either $M_i =_\beta I$ or $M_i =_\beta$ a Church numeral. In particular, we have the following lemma.

Lemma 1. *If M is a combinator such that $M\Omega\Delta =_\beta \Delta$ and $MI =_\beta I$, then either $M =_\beta I$ or $M =_\beta$ some Church numeral.*

Proof. Since $MI =_\beta I$, M is solvable with head normal form say $\lambda x_1 \ldots x_n.x_iX_1 \ldots X_m$ with $n \leqslant 2$. If $n = 1$, we have

$$M\Omega\Delta \twoheadrightarrow \Omega[\Omega/x_1]X_1 \ldots [\Omega/x_1]X_m\Delta \underset{\beta}{=} \Delta.$$

Since no reduct of Δ has more than three components, we have $m = 0$ so $M =_\beta I$. Thus we may assume $n = 2$.

Case 1: $i = 2$. Then,

$$MI \twoheadrightarrow \lambda x_2x_2[I/x_1]X_1 \ldots [I/x_1]X_m \twoheadrightarrow I,$$

so $m = 0$ and $M =_\beta \underline{0}$.

Case 2: $i = 1$. Then,

$$M\Omega\Delta \twoheadrightarrow \Omega([\Omega/x_1, \Delta/x_2]X_1) \ldots ([\Omega/x_1, \Delta/x_2]X_m) \underset{\beta}{=} \Delta,$$

so, arguing as above, $m = 1$. Since $\Omega([\Omega/x_1, \Delta/x_2]X_1) =_\beta \Omega\Delta$, we have $\lambda x_1x_2X_1.\Omega\Delta = \Delta$. In addition, since

$$\lambda x_1x_2x_1X_1.I \rightarrow \lambda x_2I([I/x_1]X_1) \rightarrow \lambda x_2[I/x_1]X_1 \twoheadrightarrow I,$$

we have $\lambda x_1x_2X_1.I =_\beta I$. Thus, $\lambda x_1x_2X_1$ satisfies the same conditions as M in the case $n = 2$. Repeating the above either terminates in the case $i = 2$ and so $\exists n$: $M =_\beta \underline{n}$,

$$\begin{array}{ll} \lambda x_1 x_2 & x_1 \\ & | \\ & x_1 \\ & | \\ & x_1 \\ & \vdots \end{array}$$

Fig. 2.

or it fails to terminate. In the latter case, M has the infinite Böhm tree in Fig. 2. Then, however, MI is unsolvable contradicting the choice of M. $\square$

Let M^* be the result of a complete development of M.

Lemma 2. *If $E_n M_1 \ldots M_m =_\beta E_p$, then, for $1 \leq i \leq m$, either $M_i =_\beta I$ or $M_i =_\beta$ a Church numeral*

Proof. Consider the following two reduction sequences:

(1) The cofinal reduction sequence for E_p:

$$E_p \twoheadrightarrow A_k \equiv \lambda z_1 \ldots z_k X_k Y_k Z_k,$$

(where $X_k \equiv \theta(\lambda u \lambda xyz\, u\, U(y([z\Omega V, zI])))$ and $Pxz \twoheadrightarrow U$ and $\Delta \twoheadrightarrow V$ and

$$P(\ldots(P\underline{p}z_1)\ldots)z_k \twoheadrightarrow Y_k \quad \text{and} \quad \Xi(\Gamma z_1)\ldots(\Gamma z_k) \twoheadrightarrow Z_k$$

$$A_k \twoheadrightarrow \lambda z_1 \ldots z_k(\lambda v\, v(\theta v))(\lambda u \lambda xyz\, u\, U^*(y([z\Omega V^*, zI])))\, Y_k^* Z_k^*$$

by a complete development

$$\begin{aligned} &\twoheadrightarrow A_{k+1} \\ &\qquad \equiv \lambda z_1 \ldots z_{k+1}\theta(\lambda u \lambda xyz\, u\, U^*(y([z\Omega V^*, zI])))([Y_k^*/x, z_{k+1}/z](U^*)) \\ &\qquad\qquad (Z_k^*[z_{k+1}\Omega V^*, z_{k+1}I])) \\ &\twoheadrightarrow \cdots. \end{aligned}$$

(2) the 'head' reduction sequence for $E_n M_1 \ldots M_m$

$$\begin{aligned} E_n M_1 \ldots M_m &\twoheadrightarrow (\lambda z_1 \ldots z_m E(P(\ldots(P\underline{n}z_1)\ldots)z_m)(\Xi(\Gamma z_1)\ldots(\Gamma z_m)))M_1 \ldots M_m \\ &\twoheadrightarrow \alpha_0 \equiv E\underbrace{(P(\ldots(P\underline{n}M_1)\ldots)M_m)}_{\beta_0}\underbrace{(\Xi(\Gamma M_1)\ldots(\Gamma M_m))}_{\gamma_0} \\ &\xrightarrow{\text{head}}\!\!\twoheadrightarrow \lambda z_1 \ldots z_q E\underbrace{(P(\ldots(P\beta_0 z_1)\ldots)z_q)}_{\beta_q}\underbrace{(\gamma_0(\Gamma z_1)\ldots(\Gamma z_q))}_{\gamma_q} \equiv \alpha_q \end{aligned}$$

$$\xrightarrow{\text{head}} \lambda z_1 \ldots z_q \lambda vv(\theta v)\Psi\beta_q\gamma_q$$

$$\xrightarrow{\text{head}} \lambda z_1 \ldots z_q \Psi(\theta\Psi)\beta_q\gamma_q$$

$$\xrightarrow{\text{head}} \lambda z_1 \ldots z_q(\lambda xyzE(Pxz)(y(\Gamma z)))\beta_q\gamma_q$$

$$\xrightarrow{\text{head}} \lambda z_1 \ldots z_q(\lambda yzE(P\beta_q z)(y(\Gamma z)))\gamma_q$$

$$\xrightarrow{\text{head}} \lambda z_1 \ldots z_{q+1}E\underbrace{(P\beta_q z_{q+1})}_{\beta_{q+1}}\underbrace{(\gamma_q(\Gamma z_{q+1}))}_{\gamma_{q+1}} \equiv \alpha_{q+1}$$

$$\xrightarrow{\text{head}}\!\!\!\!\rightarrow \cdots.$$

If $E_n M_1 \ldots M_m =_\beta E_p$, then, for all sufficiently large k, $\alpha_0 \twoheadrightarrow A_k$. Consider a standardizing reduction sequence

$$\alpha_0 \underset{\text{head}}{\twoheadrightarrow} N \underset{\text{internal}}{\twoheadrightarrow} A_k.$$

Now A_k has a matrix consisting of five components (the first two of which comprise θ). Thus, $N \equiv \alpha_q$ for some q, and $q = k$. Thus,

$$\alpha_k \overset{\text{internal}}{\twoheadrightarrow} A_k,$$

so $\beta_k \twoheadrightarrow Y_k$ and $\gamma_k \twoheadrightarrow Z_k$.

In particular, $\Xi(\Gamma M_1) \ldots (\Gamma M_m) =_\beta \Xi$, so, for $1 \leq i \leq m$, $M_i\Omega\Delta =_\beta \Delta$ and $M_i I =_\beta I$. Thus, by Lemma 1, for $1 \leq i \leq m$, either $M_i =_\beta I$ or $M_i =_\beta$ a Church numeral. □

Lemma 3. *If $E_n M_1 \ldots M_m =_\beta E_p$, then $n \sqsubseteq p$.*

Proof. By Lemma 2 we know that $M_i =_\beta I$ or $M_i =_\beta$ a Church numeral. Referring to the proof of Lemma 2 we also know that $\beta_k \twoheadrightarrow Y_k$, so $\beta_k =_\beta P(\ldots(P\underline{p}z_1)\ldots)z_k$. Thus,

$$\underline{p} = \underbrace{P(\ldots(P\underline{p}\underline{p})\ldots)\underline{p}}_{k} =_\beta P(\ldots(P(P(\ldots(P\underline{n}M_1)\ldots)M_m)\underbrace{\underline{p})\ldots)\underline{p}}_{k}.$$

By the choice of P we may assume that, for $1 \leq i \leq m$, $M_i =_\beta \underline{q}_i$. Thus, $\underline{p} =_\beta P(P(\ldots(P\underline{n}\underline{q}_1)\ldots)\underline{q}_m)\underline{p}$, so $p \sqsupseteq n$.

Now, if $n \sqsubseteq p$, then

$$\begin{aligned} E_n \underline{p} &=_\beta \lambda z_2 E(P(P\underline{n}\underline{p})z_2)(\Xi(\Gamma\underline{p})(\Gamma z_2)) \\ &=_\beta \lambda z_2 E(P\underline{p}z_2)(\Xi[\Delta, I](\Gamma z_2)) \\ &=_\beta \lambda z_2 E(P\underline{p}z_2)(\Xi(\Gamma z_2)) =_\beta E_p, \end{aligned}$$

so $E_n \leqslant E_p$. This completes the proof that the map $n \mapsto E_n$ is an isomorphic embedding. $\square$

References

[1] H.P. Barendregt, *The Lambda Calculus* (North-Holland Amsterdam, 1984).
[2] A. Mostowski, *Foundational Studies: Selected Works Vol. II* (North-Holland, Amsterdam, 1979) 68–70.